EFFICIENT COMPUTATION OF THE INFIMUM IN H_∞ CONTROL FOR SEISMIC STRUCTURES

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Abstract

An important consideration in the design of H_∞ controllers is the optimal norm of the H_∞ control problem. This value determines the lowest value of the H_∞ norm that can be obtained with the problem and system defined. Hence, it represents a design limit, but one that is computationally intractable and difficult to obtain. A new method for determining the optimal H_∞ norm of a state feedback system is presented. It is based on the application of discriminant to check a stability condition on the Hamiltonian matrix that is associated with the infimum value. In addition, a generalized eigenvalue problem is deduced from the discriminant stability condition to avoid any required iteration. The overall approach provides a highly accurate approximation of the optimal value with minimum computation compared to other approaches in the literature.

Introduction

H_∞ control limits the infinity norm of the transfer function between the disturbance inputs and regulated outputs to a specific value γ. Many computational considerations in the design of H_∞ controllers require the determination of the optimal H_∞ norm, or the infimum of the H_∞ optimal control problem (denoted γ^* in this paper). Computationally, intense to evaluate, this value represents the smallest possible γ value that can be obtained with the given H_∞ problem definition. It is thus a measure of the ability of feedback control to modify this system dynamic with the given feedback problem definition provided.

The computation of this infimum has normally been studied based on either iterative or non-iterative methods. The iterative method typically includes the algebraic Riccati equation (ARE) approach (Doyle *et al*., 1989; Scherer, 1990; Lin *et al*., 1999) and the linear matrix inequality (LMI) approach (Stoorvogel, 1992; Gahinet and Apkarian, 1994). Both of these iterative approaches usually start with a given $\gamma > 0$ and test whether $\gamma > \gamma^*$. An iterative scheme is thus employed to find the infimum γ^* by repeating this test for different values of γ. These algorithms are computationally very expensive due to the potentially numerous iterations requiring an ARE or LMI problem solution. In contrast, non-iterative methods (Chen *et al*., 1992a; Chu, 2004) can directly compute the infimum, γ^*, without iteration. However, certain intricate transformations are necessary to first transform the original problem of computing γ^* into mathematically feasible forms. These transformations are followed by the solution of an ARE, a Lyapunov equation and an eigenvalue problem. Accordingly, the computational efficiency may not be significantly better with non-iterative algorithms.

In a very recent paper by the authors (Wu *et al*., 2006), a novel iterative algorithm was developed for the determination of γ^*. First, the eigenvalues of the Hamiltonian matrix associated with the H_∞ ARE problem were examined to define a borderline stability criterion by the occurrence of pure imaginary eigenvalues when the infimum is approached. Based on this stability criterion, the classical Routh-Hurwitz theorem was employed to check the system stability, requiring only the characteristic polynomial coefficients of the Hamiltonian matrix for any given value of γ. Moreover, it was proved that the characteristic polynomial can be analytically expressed in terms of γ and thus used to economically obtain the polynomial coefficients corresponding to various values of γ required in the iteration process. With the combination of these ingredients, a Routh-Hurwitz method to compute the optimal H_∞ norm was then established for the state feedback problems.

Extended from the previous work, a more efficient non-iterative method is further developed in this study with the application of the polynomial discriminant. An alternative stability borderline check is obtained from the interesting occurrence of double roots at the stability threshold. This multiple eigenvalue criterion leads to a more compact discriminant formulation, with which a generalized eigenvalue problem can be deduced for the direct determination of γ^* without iteration. Certain numerical issues are finally discussed with a couple of numerical examples in structural control problems.

Problem Statement and Stability Criterion

Problem Statement

Consider the standard linear time-invariant system defined:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{Ew} \\ \mathbf{z} &= \mathbf{Cx} + \mathbf{Du}\end{aligned} \tag{1}$$

where the state $\mathbf{x} \in R^n$, the control input $\mathbf{u} \in R^m$, the disturbance $\mathbf{w} \in R^l$, and the regulated output $\mathbf{z} \in R^p$. In addition, **A**, **B**, **E**, **C** and **D** are constant matrices of appropriate dimension. The H_∞ norm of this system S is defined in the time domain as

$$\|S\|_\infty = \sup_{\mathbf{w}} \frac{\|\mathbf{z}\|_2}{\|\mathbf{w}\|_2} \tag{2}$$

If state feedback, $\mathbf{u} = \mathbf{Gx}$, is considered, the infimum of the H_∞ norm for S can be defined:

$$\gamma^* = \inf\left\{ \|S\|_\infty \;\middle|\; \mathbf{G} \in R^{m\times n}, \mathbf{A}_{cl} \text{ is stable} \right\} \tag{3}$$

where $\mathbf{A}_{cl}$ is the closed-loop plant matrix. In other words, the optimal H_∞ norm γ^* is the minimum γ value for which controlled system stability can be guaranteed.

For a given suboptimal $\gamma > \gamma^*$, the corresponding H_∞ control problem is to determine the state feedback control gain matrix **G** such that $\|S\|_\infty < \gamma$. To mathematically analyze the H_∞ control problem, a quadratic performance index J is usually defined to reformulate the problem as:

$$\min_{\mathbf{u}}\max_{\mathbf{w}} J = \min_{\mathbf{u}}\max_{\mathbf{w}} \frac{1}{2}\int_0^\infty \left[\mathbf{z}^{\mathrm{T}}\mathbf{z} - \gamma^2 \mathbf{w}^{\mathrm{T}}\mathbf{w}\right] \mathrm{d}t < 0 \tag{4}$$

Calculus of variation has been applied to solve this optimization problem and the resulting solution takes the form of an ARE (Doyle *et al.*, 1989). In practical applications, it is normally further assumed that $\mathbf{C}^{\mathrm{T}}\mathbf{D} = \mathbf{0}$ and $\mathbf{D}^{\mathrm{T}}\mathbf{D}$ is full rank. With these conditions, the original H_∞ ARE can be simplified to

$$\mathbf{PA}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\left[\frac{1}{\gamma^2}\mathbf{EE}^{\mathrm{T}}-\mathbf{B}\left(\mathbf{D}^{\mathrm{T}}\mathbf{D}\right)^{-1}\mathbf{B}^{\mathrm{T}}\right]\mathbf{P}+\mathbf{C}^{\mathrm{T}}\mathbf{C}=\mathbf{0} \tag{5}$$

where $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$ is the Riccati matrix. To obtain this $n \times n$ Riccati matrix **P**, it is most convenient to transform (5) into a linear eigenvalue problem for the Hamiltonian matrix, **H**, defined:

$$\mathbf{H}=\begin{bmatrix} \mathbf{A} & \frac{1}{\gamma^2}\mathbf{EE}^{\mathrm{T}}-\mathbf{B}\left(\mathbf{D}^{\mathrm{T}}\mathbf{D}\right)^{-1}\mathbf{B}^{\mathrm{T}} \\ -\mathbf{C}^{\mathrm{T}}\mathbf{C} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \tag{6}$$

Thus, **P** can be directly obtained from the eigenvalues and eigenvectors of **H** (Meirovitch, 1990).

Stability Criterion and Borderline

It is clear that all the eigenvalues of the closed-loop system matrix $\mathbf{A}_{cl}$ are included in those of **H** with the transformation from (5) to (6) (Meirovitch, 1990). Moreover, Potter (1966) also proved that the eigenvalues of **H** appear in anti-symmetric pairs $\pm\lambda_1, \pm\lambda_2, \cdots\cdots, \pm\lambda_n$ in the complex plane. Following from these attributes, there are only n eigenvalues with negative real parts for **H** and these stable eigenvalues have to be selected in the determination of **P** to find a stable closed-loop solution for a given value of γ. In addition, these eigenvalues move in the complex plane as the value of γ is changed, while retaining the anti-symmetry in the complex plane. Therefore, for a stable closed-loop solution, no value of γ can be chosen that results in pure imaginary, borderline stable eigenvalues of **H**. Thus, a simple stability criterion for the H_∞ infimum can be established by prohibiting values of γ for which eigenvalues of **H** are located on the imaginary axis of the complex plane. The infimum, γ^*, is the value of γ where the first eigenvalues become purely imaginary valued and meet the imaginary axis.

The above concept is further explored in this study. Since the eigenvalues of **H** appear in pairs with opposite signs, all the possible eigenvalues of **H** for a stable controlled system must be either in the form of real pairs with opposite signs or in the form of complex conjugate quartets as $\pm\alpha_j \pm \mathrm{i}\beta_j$ where $\alpha_j \neq 0$, $\beta_j \neq 0$ and $\mathrm{i}=\sqrt{-1}$. The real parts of at least one group of eigenvalues would vanish and reach the imaginary axis in the complex plane when the stability borderline is reached. If this starts with one real pair with opposite signs, the two eigenvalues would coincide at the origin of the complex plane and become a double root with a value of zero. On the other hand, two double roots with opposite pure imaginary values would be induced if this happens to a complex conjugate quartet. An alternative stability borderline check can be naturally deduced from the interesting occurrence of double roots at the stability threshold. With the condition that the open-loop system is stable and without multiple co-located eigenvalues, the infimum is the value of γ where the first multiple eigenvalues occur.

Non-iterative Discriminant Method

The discriminant of a polynomial provides a convenient procedure to discriminate the existence of multiple roots merely from the coefficients of a general polynomial. According to the new stability criterion established in the previous section, it can be consequently implemented to check the stability of a H_∞ controlled system based on the characteristic polynomial of its Hamiltonian matrix.

Sylvester Matrix and Discriminant

For two polynomials $f(x)$ of degree d_1 and $g(x)$ of degree d_2, the Sylvester matrix associated with $f(x)$ and $g(x)$ is an $(d_1+d_2)\times(d_1+d_2)$ matrix. It is formed by filling the matrix with the coefficients of $f(x)$ beginning with the upper left corner, then shifting down one row and one column to the right and filling in the coefficients starting there until they hit the right side. This process is then repeated for the coefficients of $g(x)$. The determinant of the Sylvester matrix of two polynomials is called the resultant of the polynomials. In addition, the discriminant of a polynomial $h(x)$ is defined as the resultant of $h(x)$ and its first derivative $h'(x)$. It can be shown that a polynomial has at least one multiple root if and only if its discriminant equals to zero (Cohen, 1993).

Considering the fact that its roots exist in anti-symmetric pairs with opposite signs, the characteristic polynomial of **H** is a function of λ^2 with real valued coefficients and is expressed as

$$p(\lambda)=\lambda^{2n}+a_{2n-2}\lambda^{2n-2}+\cdots\cdots+a_2\lambda^2+a_0 \tag{7}$$

with its derivative $p'(\lambda)$ given by

$$p'(\lambda)=2n\lambda^{2n-1}+(2n-2)a_{2n-2}\lambda^{2n-3}+\ldots+2a_2(\overline{\gamma})\lambda \tag{8}$$

With the coefficients in (7) and (8), the discriminant of $p(\lambda)$ can then be evaluated by a $(4n-1)\times(4n-1)$ determinant.

Stability Borderline Check

With any given value of γ and constant matrices **A**, **B**, **C**, **D** and **E**, the corresponding Hamiltonian matrix **H** under H_∞ control is constructed from (6). The characteristic polynomial $p(\lambda)$ corresponding to **H**, in the form of (7), can then be computed from effective algorithms. Based on the coefficients for $p(\lambda)$, the discriminant $D_{p(\lambda)}$ of this polynomial is easily evaluated by performing a matrix determinant operation. Thus, the calculated value of $D_{p(\lambda)}$ would become a convenient index to check the stability borderline of the controlled system. More specifically, the system stability borderline is reached when $D_{p(\lambda)}=0$.

In general, the discriminant method evaluates one determinant and can be regarded as a more compact and organized version of the Routh-Hurwitz method where a series of sub-determinants need to be computed. It is also noteworthy that the Routh-Hurwitz method deals with the $2n\times 2n$ Hamiltonian matrix **H** of a controlled system, instead of the $n\times n$ closed-loop system matrix $\mathbf{A}_{cl}$ in a direct way. The interesting fact is that the dimension of the target matrix (from $\mathbf{A}_{cl}$ to **H**) has to be doubled for mirroring the eigenvalues in the complex plane such that the correspondence between instability and the occurrence of pure imaginary or multiple eigenvalues can be established. For applying the discriminant method, the $2n\times 2n$ Hamiltonian matrix, **H**, needs to be further expanded to a $(4n-1)\times(4n-1)$ Sylvester matrix for examining root multiplicity. Even though the dimension of the target matrix is increased almost by two with the discriminant method, this problem can be eliminated by the following reduction in computation.

Reduction of Degree

Since the characteristic polynomial $p(\lambda)$ of **H** is a function of λ^2 with degree $2n$ as shown in (7), the degree of $p(\lambda)$ can be reduced by letting $\delta=\lambda^2$ and expressing the polynomial in terms of δ:

$$p(\delta) = \delta^n + a_{2n-2}\delta^{n-1} + \cdots\cdots + a_2\delta + a_0 \tag{9}$$

However, the different features for the roots of λ and $\delta = \lambda^2$ need to be carefully investigated before employing the reduced polynomial $p(\delta)$ with degree *n*.

If the control eigenvalues for system stability are in the form of complex conjugate quartets, they will appear as $\lambda_1, \lambda_2 = -\alpha \pm \mathrm{i}\beta$ together with symmetric $\lambda_3, \lambda_4 = \alpha \pm \mathrm{i}\beta$ where $\alpha \geq 0$ and $\beta > 0$. These four eigenvalues of $p(\lambda)$ correspond to two roots of $p(\delta)$ by $\delta_1 = \lambda_1^{\ 2} = \lambda_3^{\ 2} = \left(\alpha^2 - \beta^2\right) - \mathrm{i}(2\alpha\beta)$ and $\delta_2 = \lambda_2^{\ 2} = \lambda_4^{\ 2} = \left(\alpha^2 - \beta^2\right) + \mathrm{i}(2\alpha\beta)$. When $\alpha \to 0$ at the stability limit, it is found that $\delta_1 = \delta_2 = -\beta^2$, indicating a double root in $p(\delta)$ on the negative real axis. In other words, $p(\delta)$ can be conveniently adopted to replace $p(\lambda)$ for checking the occurrence of multiple roots, and hence the stability limit, under such circumstances. Therefore, the discriminant of the H_∞ control problem can be more efficiently evaluated from the reduced polynomial $p(\delta)$ by a $(2n-1)\times(2n-1)$ determinant.

Non-iterative Method with Eigenvalue Evaluation

If $\bar{\gamma} = 1/\gamma^2$ is assumed to reformulate the Hamiltonian matrix **H** given by (6), the following theorem has been proved by Wu *et al*. (2006):

$$p(\delta) = \sum_{i=0}^{n} a_{2i}(\bar{\gamma})\delta^i \tag{10}$$

where the coefficients $a_{2i}(\bar{\gamma})$ are expressed as polynomials in terms of $\bar{\gamma}$ with degree *r* at most and *r* is the minimum rank of the matrices $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ and $\mathbf{C}^{\mathrm{T}}\mathbf{C}$. Based on this theorem, the reduced characteristic polynomial $p(\delta)$ can be further reorganized with respect to $\bar{\gamma}$ as

$$p(\delta) = p_0(\delta) + p_1(\delta)\bar{\gamma} + \cdots\cdots + p_{r-1}(\delta)\bar{\gamma}^{r-1} + p_r(\delta)\bar{\gamma}^r = \sum_{j=0}^{r} p_j(\delta)\bar{\gamma}^j \tag{11}$$

where $p_j(\delta)$ are polynomials in terms of δ with degree *n* at most. From (11), the coefficients of each polynomial $p_j(\delta)$ can be uniquely solved from $r+1$ numerical calculations of the characteristic polynomial corresponding to $r+1$ different selected values of $\bar{\gamma}$.

Another point of view to look at (11) is to consider $p(\delta)$ as a linear combination of $p_0(\delta)$, $p_1(\delta)$, $\cdots\cdots$, $p_r(\delta)$ for any given value of $\bar{\gamma}$. Accordingly, the discriminant of $p(\delta)$ can be decomposed as

$$\begin{aligned} D_{p(\delta)} &= D_{p_0(\delta)+p_1(\delta)\bar{\gamma}+\cdots\cdots+p_{r-1}(\delta)\bar{\gamma}^{r-1}+p_r(\delta)\bar{\gamma}^r} \\ &= \left|\mathbf{P}_0 + \bar{\gamma}\,\mathbf{P}_1 + \cdots\cdots + \bar{\gamma}^{r-1}\mathbf{P}_{r-1} + \bar{\gamma}^r\mathbf{P}_r\right| \end{aligned} \tag{12}$$

where $\mathbf{P}_0$, $\mathbf{P}_1$, $\cdots\cdots$, $\mathbf{P}_r$ are the Sylvester matrices of $p_0(\delta)$, $p_1(\delta)$, $\cdots\cdots$, $p_r(\delta)$, respectively. It should be noted that the degree of $p_j(\delta)$ in terms of δ may be less than *n*. In this case, superfluous zero coefficients need to be added in the highest few terms of polynomial such that $p_0(\delta)$, $p_1(\delta)$, $\cdots\cdots$, $p_r(\delta)$ are all in the form of *n*-th degree polynomial and $\mathbf{P}_0$, $\mathbf{P}_1$, $\cdots\cdots$, $\mathbf{P}_r$ are all of the same

$(2n-1)\times(2n-1)$ dimension. Since the system stability borderline is reached when $D_{p(\delta)}=0$, (12) clearly indicates that the optimal control norm γ^* can be solved from a generalized eigenvalue problem:

$$\left|\mathbf{P}_0+\bar{\gamma}\,\mathbf{P}_1+\cdots\cdots+\bar{\gamma}^{r-1}\mathbf{P}_{r-1}+\bar{\gamma}^{r}\mathbf{P}_r\right|=0 \tag{13}$$

More specifically, considering that the infimum γ^* has to be positive, $\bar{\gamma}^*=\left(\gamma^*\right)^{-2}$ can first be conveniently obtained by selecting the minimum positive eigenvalue of (13). The determination of γ^* is then easily completed by $\gamma^*=\left(\bar{\gamma}^*\right)^{-1/2}$. The above approach successfully transforms the whole solution process into a generalized eigenvalue problem of (14) to avoid any required iteration.

Numerical Examples

Example 1: SDOF Structural Control Case

A typical SDOF structural system is first taken as a demonstrative example to illustrate the non-iterative discriminant method developed in this study. With mass m, stiffness k, and damping c, its natural frequency and damping ratio are defined: $\omega=\sqrt{(k/m)}$ and $\xi=c/(2m\omega)$. If **C** and **D** are defined corresponding to the H_∞ energy control case as in Wu and Lin (2004):

$$\mathbf{C}=\begin{bmatrix}0 & 0\\ 0 & \sqrt{\beta m}\\ 0 & 0\end{bmatrix} \quad\text{and}\quad \mathbf{D}=\begin{bmatrix}0\\ 0\\ 1/\sqrt{k}\end{bmatrix} \tag{14}$$

where β is a user specified energy weighting parameter. The associated Hamiltonian matrix **H** can then be obtained and its characteristic determinant is a fourth-degree polynomial in λ, or a second-degree polynomial in δ:

$$\begin{aligned}p(\lambda)&=\lambda^4+\omega^2\left[2-4\xi^2+\beta(\bar{\gamma}-1)\right]\lambda^2+\omega^4\\ &=\left[\delta^2+\omega^2\left(2-4\xi^2-\beta\right)\delta+\omega^4\right]+\left(\beta\omega^2\delta\right)\bar{\gamma}\\ &=p_0(\delta)+p_1(\delta)\bar{\gamma}\end{aligned} \tag{15}$$

From (15), the eigenvalue problem of (13) associated with the discriminant of this problem can be formulated as:

$$\begin{aligned}\left|\mathbf{P}_0+\bar{\gamma}\,\mathbf{P}_1\right|&=\left|\begin{bmatrix}1 & \omega^2\left(2-4\xi^2-\beta\right) & \omega^4\\ 2 & \omega^2\left(2-4\xi^2-\beta\right) & 0\\ 0 & 2 & \omega^2\left(2-4\xi^2-\beta\right)\end{bmatrix}+\bar{\gamma}\begin{bmatrix}0 & \beta\omega^2 & 0\\ 0 & \beta\omega^2 & 0\\ 0 & 0 & \beta\omega^2\end{bmatrix}\right|\\ &=-\omega^4\left[\left(2-4\xi^2\right)+\beta(\bar{\gamma}-1)\right]^2+4\omega^4=0\end{aligned} \tag{16}$$

Therefore, the two eigenvalues solved from (16) are $\bar{\gamma}=\left(\beta+4\xi^2\right)/\beta$ and $\bar{\gamma}=\left(\beta+4\xi^2-4\right)/\beta$ and the minimum positive choice of them leads to

$$\gamma^* = \sqrt{\frac{\beta}{\beta + 4\xi^2}} \tag{17}$$

(20) exactly matches what derived by the Routh-Hurwitx method (Wu *et al*. 2006).

Example 2: 8-DOF Structural Control Case

A second structural control case consisting of an eight-story shear building was considered in Wu *et al*. (2006) to investigate more complicated situations and is repeated here with the non-iterative discriminant method. Each floor is assumed to have an identical mass of 345.6 tons and a horizontal column stiffness of 340,400 kN/m. These values result in a first-mode frequency of 0.921 Hz. The damping coefficient of each floor is also taken as 2,937 tons/sec, corresponding to a first-mode damping ratio of 2.5%. All the matrices required in the Hamiltonian matrix of (6) are defined:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{8\times8} & \mathbf{I}_{8\times8} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\overline{\mathbf{C}} \end{bmatrix};\ \mathbf{C}^{\mathrm{T}}\mathbf{C} = \begin{bmatrix} \mathbf{0}_{8\times8} & \mathbf{0}_{8\times8} \\ \mathbf{0}_{8\times8} & \mathbf{M} \end{bmatrix};\ \mathbf{D}^{\mathrm{T}}\mathbf{D} = 1;\ \mathbf{B} = \begin{bmatrix} \mathbf{0}_{8\times1} \\ -\mathbf{M}^{-1}\overline{\mathbf{B}} \end{bmatrix};\ \mathbf{E} = \begin{bmatrix} \mathbf{0}_{8\times1} \\ -\mathbf{M}^{-1}\overline{\mathbf{E}} \end{bmatrix} \tag{18}$$

where **0** denotes the zero matrix and the subscripts specify the associated matrix dimension. In (18), the corresponding mass, stiffness and damping matrices are:

$$\mathbf{M} = 345.6\mathbf{I}_{8\times8};\ \mathbf{K} = 340400\mathbf{T}_{8\times8};\ \overline{\mathbf{C}} = 2937\mathbf{T}_{8\times8} \tag{19}$$

where

$$\mathbf{T}_{8\times8} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{20}$$

In addition, if the earthquake problem with a single excitation input is considered and one control input is exerted on the top floor, as from an active mass damper, the matrices $\overline{\mathbf{B}}$ and $\overline{\mathbf{E}}$ in (18) are defined:

$$\overline{\mathbf{B}} = \begin{bmatrix} 1 \\ \mathbf{0}_{7\times1} \end{bmatrix} \text{ and } \overline{\mathbf{E}} = \mathbf{1}_{8\times1} \tag{21}$$

where **1** is the matrix with all the elements of ones.

In this example, $\operatorname{rank}\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}\right) = 1$ and $\operatorname{rank}\left(\mathbf{C}^{\mathrm{T}}\mathbf{C}\right) = 8$. It is clear that the coefficients of the characteristic polynomial of **H** given by (11) will be linear in $\bar{\gamma}$, defined:

$$p(\delta) = p_0(\delta) + p_1(\delta)\bar{\gamma} \tag{22}$$

The coefficients of $p_0(\delta)$ and $p_1(\delta)$ are calculated by computing two sets of the characteristic polynomial coefficients in Matlab for two values of $\bar{\gamma}=0$ and $\bar{\gamma}=1$. $\mathbf{P}_0$ and $\mathbf{P}_1$ can then be constructed from $p_0(\delta)$ and $p_1(\delta)$. The corresponding eigenvalue problem is subsequently solved to obtain all the candidates for $\bar{\gamma}^*$. The minimum positive eigenvalue in this case is find to be $\bar{\gamma}^*=8.484137$ and leads to $\gamma^*=0.3433177$. This approximation, compared to the corresponding iterative solution $\gamma^*=0.3433175$, is accurate to within an absolute relative percentage error of 6×10^{-7}.

Conclusions

A new method for determining the optimal H_∞ norm, or infimum, of a closed-loop system has been developed and presented. The new method is computationally far less intense as it does not require any iteration. This method is based on the application of discriminant to check a stability condition on the Hamiltonian matrix that is associated with the infimum value. In addition, a generalized eigenvalue problem is deduced from the discriminant stability condition. As a result, the approach provides the desired result with minimum computation compared to other approaches in the literature. Two test cases are presented and errors in the approximate solution versus published results and iterative eigenvalue solutions are within 6×10^{-7}. Overall, the methods and theory presented comprise a discriminant based semi-analytical approach for determining the H_∞ norm infimum of a control system, and are a significant step forward in this area of work.

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References

Chen, B. M., A. Saberi and U. Ly (1992), "A Non-iterative Method for Computing the Infimum in H_∞-optimisation", *International Journal of Control*, **56**, 1399-1418.

Cohen, H. (1993), *A Course in Computational Algebraic Number Theory*, Springer-Verlag, New York.

Chu, D. (2004), "On the Computation of the Infimum in H_∞-optimization", *Numerical Linear Algebra Appl.*, **11**, 619-648.

Doyle, J. C., K. Glover, P. P. Khargonekar and B. A. Francis (1989), "State-space Solutions to Standard H_2 and H_∞ Control Problems", *IEEE Transactions on Automatic Control*, **34**, 831-847.

Gahinet, P. and P. Apkarian (1994), "A Linear Matrix Inequality Approach to H_∞ Control", *International Journal of Robust Nonlinear Control*, **4**, 421-448.

Lin, W.-W., C.-S. Wang and Q.-F. Xu (1999), "On the Computation of the Optimal H_∞ Norms for Two Feedback Control Problems", *Linear Algebra and Its Applications*, **287**, 223-255.

Meirovitch, L. (1990), *Dynamics and Control of Structures*, Wiley, New York.

Potter, B. (1966) , "Matrix Quadratic Solutions", *SIAM Journal of Applied Mathematics*, **14,** 496-501.

Scherer, C. (1990), "H_∞-control by State-feedback and Fast Algorithms for the Computation of Optimal H_∞-norms", *IEEE Transactions on Automatic Control*, **AC-35**, 1090-1099.

Stoorvogel, A. A. (1992), *The H_∞ Control Problem: A State Space Approach*, Prentice-Hall, Englewood Cliffs.

Wu, W.-H. and C.-C. Lin (2004), "H_∞ Energy Control and Its Stability Analysis for Civil Engineering Structures", *Structural Control and Health Monitoring*, **11**, 161-187.

Wu, W.-H., J. G. Chase, C. E. Hann and J. B. Mander "A Novel Routh-Hurwitz Method to Compute the Optimal H_∞ Norm for State Feedback Problems", submitted to *International Journal of Control*.